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NOTE

Transport Equations with Quadratic Nonlinearities

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Abstract

Two papers have recently appeared with approximate analysis of transport equations with quadratic nonlinearities. These equations can generally be solved exactly by means of a linearizing transformation, although there are exceptions that depend on the type of boundary conditions.

Two papers have recently appeared in literature of separation science that deal with the solution of a nonlinear transport equation (*1*, *2*). In both of these a lengthy approximate solution was used that is valid only over a part of the range in which the equations themselves are valid. In this note we show that the equations in both Refs. *1* and *2* can be solved exactly and give some indication of the types of equations that can be solved by these techniques. We also note that several applications of the linearizing transformation have been made in earlier investigations of separation processes (*3*–*8*).

The prototype of a linear one dimensional transport equation is

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - v\frac{\partial c}{\partial x} \tag{1}$$

in which D has the dimensions of a diffusion constant and v has the dimensions of velocity. If one simply changes the dependent variable in Eq. (1) by setting $c = f(u)$, then one finds that it satisfies

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} + D\frac{d \ln f'(u)}{du}\left(\frac{\partial u}{\partial x}\right)^2 \tag{2}$$

where $f'(u) = df/du$. In a recent investigation of stop-flow sample injection in HPLC and GPC, Kubin and Vozka (*1*) derived an equation similar to Eq. (2) with $v = 0$ and

$$D\frac{d \ln f'(u)}{du} = \beta D = \text{constant} \tag{3}$$

But this ordinary differential equation is easily solved, allowing us to write

$$f(u) = \exp(\beta u) \tag{4}$$

Hence if one sets

$$u = \frac{1}{\beta} \ln c \tag{5}$$

the nonlinear equation given by Kubin and Vozka can reduce exactly to the linear heat equation and standard methods of solution can be applied. Notice that even when the coefficient of the nonlinear term in Eq. (2) is a complicated function of u, it is still simpler to solve

$$\frac{d^2 f}{du^2} = \beta(u)\frac{df}{du} \tag{6}$$

numerically followed by a solution of a linear partial differential equation than it is to solve the full nonlinear partial differential equation numerically.

A second form of nonlinear transport equation that can be solved by essentially the same technique is exemplified by the equation for separation in a cascade as recently studied by Wieck and Ishida (*2*), using an approximate theory described by Cohen (*9*). In these equations one starts from a transport equation of the form

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x} - ac(c_0 - c)\right) \tag{7}$$

in which the nonlinearity also occurs as a quadratic and a is a constant. If one makes the substitution

$$c = \frac{D}{a}\frac{1}{\psi}\frac{\partial \psi}{\partial x} \tag{8}$$

where ψ is a new independent variable, then ψ satisfies

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} - ac_0\frac{\partial \psi}{\partial x} \tag{9}$$

which is linear. Montroll and Newell have solved several cascade problems using this technique (*4*). It should be noted that although the underlying transport equations in Eqs. (2) and (7) can be linearized exactly, the boundary conditions may assume a more complicated form. It is easily verified that boundary conditions in which u (in Eq. 2) or c (in Eq. 7) are constant at given points lead to no difficulties nor do boundary conditions for which the gradient vanishes. For Eq. (7) the boundary condition corresponding to zero flux at a point

$$D\frac{\partial c}{\partial x} = ac(c_0 - c) \tag{10}$$

is also linearized by the transformation of Eq. (8). However, the general radiation condition

$$\frac{\partial c}{\partial x} + hc = 0 \tag{11}$$

can lead to nonlinearities in the boundary conditions which may be as difficult to handle as the original equation.

As a final example we note that the flow through polymer gels is often modeled by using Darcy's law (*10*). If v is the velocity of the mobile phase, P the pressure, ρ the density, and F a constant term due to whatever force provides convection, then Darcy's law provides the relationship

$$v = -k\frac{\partial P}{\partial x} + k\rho F \tag{12}$$

where k is a constant. This relation is found to hold at sufficiently slow rates of flow. This equation, together with a continuity equation, leads to an equation of the form

$$p_0\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left[\rho k\left(\frac{\partial P}{\partial x} - \rho F\right)\right] \tag{13}$$

where p_0 is the gel porosity. In order to find a single equation for ρ we must provide a constitutive relation between P and ρ. This is often taken to be

$$\rho = \rho_0 \exp\left[\beta(P - P_0)\right] \tag{14}$$

where ρ_0 and P_0 are reference values and β is the compressibility. When this last relation is assumed to hold, the density equation becomes

$$p_0\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left[k\left(\frac{\partial \rho}{\partial x} - \rho^2 F\right)\right] \tag{15}$$

which is seen to be of the form of Eq. (7) and can be reduced to linear form by using Eq. (8).

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